

A NOTE ON THE STRONG CONVERGENCE OF TWO-DIMENSIONAL WALSH-FOURIER SERIES

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ABSTRACT. The main aim of this paper is to investigate the quadratical partial sums of the two-dimensional Walsh-Fourier series.

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Let \mathbf{N}_+ denote the set of positive integers, $\mathbf{N} := \mathbf{N}_+ \cup \{0\}$. Denote by Z_2 the discrete cyclic group of order 2, that is $Z_2 = \{0, 1\}$, where the group operation is the modulo 2 addition and every subset is open. The Haar measure on Z_2 is given such that the measure of a singleton is $1/2$. Let G be the complete direct product of the countable infinite copies of the compact group Z_2 . The elements of G are of the form $x = (x_0, x_1, \dots, x_k, \dots)$ with $x_k \in \{0, 1\}$ ($k \in \mathbf{N}$). The group operation on G is the coordinate-wise addition, the measure (denote by μ) and the topology are the product measure and topology. The compact Abelian group G is called the Walsh group. A base for the neighborhoods of G can be given in the following way:

$$\begin{aligned} I_0(x) &:= G, \\ I_n(x) &:= I_n(x_0, \dots, x_{n-1}) := \{y \in G : y = (x_0, \dots, x_{n-1}, y_n, y_{n+1}, \dots)\}, \end{aligned}$$

where $x \in G$ and $n \in \mathbf{N}_+$. Denote $I_n := I_n(0)$, for $n \in \mathbf{N}$.

If $n \in \mathbf{N}$, then $n = \sum_{i=0}^{\infty} n_i 2^i$, where $n_i \in \{0, 1\}$ ($i \in \mathbf{N}$), i. e. n is expressed in the number system of base 2. Denote $|n| := \max\{j \in \mathbf{N} : n_j \neq 0\}$, that is, $2^{|n|} \leq n < 2^{|n|+1}$.

Define the variation of an $n \in \mathbf{N}$ with binary coefficients $(n_k, k \in \mathbf{N})$ by

$$V(n) = n_0 + \sum_{k=1}^{\infty} |n_k - n_{k-1}|.$$

For $k \in \mathbf{N}$ and $x \in G$ let us denote by

$$r_k(x) := (-1)^{x_k} \quad (x \in G, k \in \mathbf{N})$$

the k -th Rademacher function.

The Walsh-Paley system is defined as the sequence of Walsh-Paley functions:

$$w_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = r_{|n|}(x) (-1)^{\sum_{k=0}^{|n|-1} n_k x_k} \quad (x \in G, n \in \mathbf{N}_+).$$

The Walsh-Dirichlet kernel is defined by

$$D_n(x) = \sum_{k=0}^{n-1} w_k(x).$$

Recall that (see [9, p.7])

$$(1) \quad D_{2^n}(x) = \begin{cases} 2^n, & x \in I_n \\ 0, & x \notin I_n \end{cases}$$

and

$$(2) \quad D_{m+2^l}(x) = D_{2^l}(x) + w_{2^l}(x) D_m(x), \text{ when } m \leq 2^l.$$

Denote by $L_p(G^2)$, $(0 < p < \infty)$ the two-dimensional Lebesgue space, with corresponding norm $\|\cdot\|_p$.

The number $\|D_n\|_1$ is called n -th Lebesgue constant. Then (see [9])

$$(3) \quad \frac{1}{8}V(n) \leq \|D_n\|_1 \leq V(n).$$

The rectangular partial sums of the two-dimensional Walsh-Fourier series of a function $f \in L_1(G^2)$ are defined as follows:

$$S_{M,N}f(x, y) := \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} \widehat{f}(i, j) w_i(x) w_j(y),$$

where the numbers $\widehat{f}(i, j) := \int_{G^2} f(x, y) w_i(x) w_j(y) d\mu(x, y)$ is said to be the (i, j) -th Walsh-Fourier coefficient of the function f .

Let $f \in L_1(G^2)$. Then the dyadic maximal function is given by

$$f^*(x, y) = \sup_{n \in \mathbf{N}} \frac{1}{\mu(I_n(x) \times I_n(y))} \left| \int_{I_n(x) \times I_n(y)} f(s, t) d\mu(s, t) \right|.$$

The dyadic Hardy space $H_p(G^2)$ $(0 < p < \infty)$ consists of all functions for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

If $f \in L_1(G^2)$, then (see [14])

$$(4) \quad \|f\|_{H_1} = \left\| \sup_{k \in \mathbf{N}} |S_{2^k, 2^k} f| \right\|_1.$$

It is known [8, p.125] that the Walsh-Paley system is not a Schauder basis in $L_1(G)$. Moreover, there exists a function in the dyadic Hardy space $H_1(G)$, the partial sums of which are not bounded in $L_1(G)$. However, Simon ([10] and [11]) proved that there is an absolute constant c_p , depending only on p , such that

$$(5) \quad \frac{1}{\log^{[p]} n} \sum_{k=1}^n \frac{\|S_k f\|_p^p}{k^{2-p}} \leq c_p \|f\|_{H_p}^p,$$

for all $f \in H_p(G)$, where $0 < p \leq 1$, $S_k f$ denotes the k -th partial sum of the Walsh-Fourier series of f and $[p]$ denotes integer part of p . (For the Vilenkin system when $p = 1$ see in Gát

[2]). When $0 < p < 1$ and $f \in H_p(G)$ the author [13] proved that sequence $\{1/k^{2-p}\}_{k=1}^\infty$ in (5) can not be improved.

For the two-dimensional Walsh-Fourier series some strong convergence theorems are proved in [12] and [15]. Convergence of quadratic partial sums was investigated in details in [3, 7]. Goginava and Gogoladze [6] proved that the following result is true:

Theorem G. Let $f \in H_1(G^2)$. Then there exists absolute constant c , such that

$$(6) \quad \sum_{n=1}^{\infty} \frac{\|S_{n,n}f\|_1}{n \log^2(n+1)} \leq c \|f\|_{H_1}.$$

The main aim of this paper is to prove that sequence $\{1/n \log^2(n+1)\}_{n=1}^\infty$ in (6) is essential too. In particular, the following is true:

Theorem 1. Let $\Phi : \mathbf{N} \rightarrow [1, \infty)$ be any nondecreasing function, satisfying the condition $\lim_{n \rightarrow \infty} \Phi(n) = +\infty$. Then

$$\sup_{\|f\|_{H_1} \leq 1} \sum_{n=1}^{\infty} \frac{\|S_{n,n}f\|_1 \Phi(n)}{n \log^2(n+1)} = \infty.$$

Proof. Let

$$f_{n,n}(x, y) = (D_{2^{n+1}}(x) - D_{2^n}(x))(D_{2^{n+1}}(y) - D_{2^n}(y)).$$

It is easy to show that

$$(7) \quad \widehat{f_{n,n}}(i, j) = \begin{cases} 1, & \text{if } (i, j) \in \{2^n, \dots, 2^{n+1} - 1\}^2, \\ 0, & \text{if } (i, j) \notin \{2^n, \dots, 2^{n+1} - 1\}^2. \end{cases}$$

Applying (1) and (4) we have

$$(8) \quad \|f_{n,n}\|_{H_1} = \left\| \sup_{k \in \mathbf{N}} |S_{2^k, 2^k} f_{n,n}| \right\|_1 = \|f_{n,n}\|_1 = 1.$$

Let $2^n < k \leq 2^{n+1}$. Combining (2) and (7) we get

$$\begin{aligned} S_{k,k} f_{n,n}(x, y) &= \sum_{i=2^n}^{k-1} \sum_{j=2^n}^{k-1} w_i(x) w_j(y) = (D_k(x) - D_{2^n}(x))(D_k(y) - D_{2^n}(y)) \\ &= w_{2^n}(x) w_{2^n}(y) D_{k-2^n}(x) D_{k-2^n}(y). \end{aligned}$$

Using (3) we have

$$(9) \quad \|S_{k,k} f_{n,n}(x, y)\|_1 \geq \int_{G^2} |D_{k-2^n}(x) D_{k-2^n}(y)| d\mu(x, y) \geq cV^2(k - 2^n).$$

Let $\Phi(n)$ be any nondecreasing, nonnegative function, satisfying condition $\lim_{n \rightarrow \infty} \Phi(n) = \infty$. Since (see Fine [1])

$$\frac{1}{n \log n} \sum_{k=1}^n V(k) = \frac{1}{4 \log 2} + o(1),$$

using (8) and (9) and Cauchy-Schwarz inequality we obtain

$$\begin{aligned}
& \sup_{\|f\|_{H_1} \leq 1} \sum_{k=1}^{2^{n+1}} \frac{\|S_{k,k} f\|_1 \Phi(k)}{k \log^2(k+1)} \geq \sum_{n=2^{n+1}}^{2^{n+1}} \frac{\|S_{k,k} f_{n,n}\|_1 \Phi(k)}{k \log^2(k+1)} \\
& \geq \frac{c\Phi(2^n)}{n^2 2^n} \sum_{n=2^{n+1}}^{2^{n+1}} V^2(k-2^n) \geq \frac{c\Phi(2^n)}{n^2 2^n} \sum_{k=1}^{2^n} V^2(k) \\
& \geq c\Phi(2^n) \left(\frac{1}{n^2 2^n} \sum_{k=1}^{2^n} V(k) \right)^2 \geq c\Phi(2^n) \rightarrow \infty, \text{ when } n \rightarrow \infty.
\end{aligned}$$

Which complete the proof of Theorem 1. \square

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